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A homotopy interior point method for semi-infinite programming problems

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Abstract This paper presents a homotopy interior point method for solving a semi-infinite programming (SIP) problem. For algorithmic purpose, based on bilevel strategy, first we illustrate appropriate necessary conditions for a solution in the framework of standard nonlinear programming (NLP), which can be solved by homotopy method. Under suitable assumptions, we can prove that the method determines a smooth interior path $\Gamma_{w^{(0)}} \subset (X^0 \times \mathcal{Y}^0) \times \Re_{++} \times \Re^l_{++} \times (0, 1]$ from a given interior point $w^{(0)} \in (X^0 \times \mathcal{Y}^0) \times \Re_{++} \times \Re^l_{++}$ to a point w^*, at which the necessary conditions are satisfied. Numerical tracing this path gives a globally convergent algorithm for the SIP. Lastly, several preliminary computational results illustrating the method are given.

Keywords Semi-infinite programming · First-order necessary optimality condition · Homotopy method · Global convergence

1 Introduction

Semi-infinite programming is an exciting part of mathematical programming, which is characterized by a finite number of variables and an infinite number of constraints. It has a wide range of applications, e.g., in engineering design, in approximation theory, in optimal control, in probability distributions. Furthermore, in finite optimization with uncertainty about parameters y from a fixed set $\mathcal{Y}$, the worst-case formulation of inequality constraints gives rise to a standard semi-infinite problem. If the set of uncertain parameters is state-dependent, i.e. $\mathcal{Y} = \mathcal{Y}(x)$, then the worst-case formulation takes the form of generalized semi-infinite programming (GSIP). Due to its wide

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applications, the study on the theory and numerical solution has been a very active research area in the last two decades. For more details of these and other applications, we refer to the conference proceedings [11] and the references cited therein, and also to the comprehensive survey [6] which covers theory, methods and applications of SIP.

The growing interest in SIPs over recent years has resulted in various contributions on the structure of the feasible set [12] and on first and second-order optimality conditions [8,12] (see the monograph [14] for an overview). To date, variously numerical approaches have been proposed to tackle problems of this kind, in which how to handle the infinite set X of constraints is one of the main features. For doing this, a commonly used way is to replace X by a finite set, and then to solve a standard finite program. In this context several approaches are well known. First, so-called Reduction Ansatz, developed by Wetterling, is to describe the feasible set of the SIP locally by finitely many inequality constraints. Hereby, the SIP program can be locally reduced to a finite one, at least conceptually (see [15]). Discretization method, the idea is to minimize its objective function subject to only a finite subset X_k of X with $X_k \subset X_{k+1}$ (usually X_k, $k = 0, \ldots, \mathcal{N}$ are grids of points), and possibly to repeat the procedure for an enlarged set X_{k+1}. The solution of SIP is approximated by the solution on the final subset $X_{\mathcal{N}}$ (see [6,11] for more detailed explanation). Exchange method, which is often more efficient than a pure discretization method, can be regarded as a compromise between discretization method and continuous reduction approach, etc. (see also [11,13]). So far, many results exhibit rapidly (locally) convergent algorithms such as Newton-like method. Then how to find good initial points is highly important. In order to derive such a point, usually one has to solve a discretization of the original problems, and this has led to the two (or even three) phase methods. Thus it is necessary to construct globally convergent algorithms. To our knowledge, the global algorithms for the SIP have those proposed by Coope and Watson [3], Watson [16] and Conn and Gould [4]. In this paper, based on local reduction and bilevel strategies, we will propose a homotopy method for the SIP, which has a global convergence.

Homotopy method, known as a class of important globally convergent method, has become a powerful tool in finding solutions of nonlinear systems (e.g. [2]). It has been given to constructively prove existence of solution and served as implementable algorithms. For some optimization problems, this method avoids the assumptions that the logarithmic barrier functions are strictly convex, in comparison with some interior-point methods. By now, this method has been further studied [5] (see also [17] for a survey discussion). However, the application of homotopy method to SIP is relatively small (see [7]). In this work, we also discuss about the feasibility of homotopy algorithm for the SIP.

To make the homotopy method available for our SIP, we reformulate the SIP into the two-level Karush–Kuhn–Tucker (KKT) system based on local reduction technique. Then, we embed the latter into a homotopy equation, which links the problem considered to a simpler one. By the parametrized Sard theorem, for almost every point of $(X^0 \times \mathcal{Y}^0) \times \Re_{++} \times \Re^l_{++}$, we obtain a path starting from that point and make the path lie in $(X^0 \times \mathcal{Y}^0) \times \Re_{++} \times \Re^l_{++} \times (0, 1]$. As a key to the homotopy method, we can prove that the path leads to a KKT point to the problem considered. A simple predictor–corrector algorithm is given to trace numerically such a path. Finally, several numerical results are given to illustrate the method.

The organization of the paper is as follows. In Sect. 2, we reformulate SIP into a two-level KKT system under suitable assumptions. In Sect. 3, we give the homotopy

and prove in details the existence of the smooth path from a given point to a KKT point to the considered problem under some "normal cone" condition. Finally, in Sect. 4, we give several preliminary numerical results.

2 Problem formulation and optimality conditions

Consider the semi-infinite programming problem in the following form:

$$\min \ f(x) \quad \text{s. t.} \quad g(x,y) \le 0, \quad \forall\, y \in \mathcal{Y} \tag{2.1}$$

with

$$\mathcal{Y} = \{y \in \Re^m : \ h_\ell(y) \le 0, \ \ell \in L\},$$

where all defining functions f, g and h_ℓ, $\ell \in L = \{1,\ldots,l\}$, are assumed to be real-valued and twice continuously differentiable on their respective domains. The index set $\mathcal{Y} \subseteq \Re^m$ is a nonempty compact subset of $\Re^m$, which may contain infinitely many elements. That is why the SIPs are called SIP problems, moreover, if the cardinality of $\mathcal{Y}$ is finite, then the SIP becomes a standard nonlinear program in the literature. For the simplicity of notation, by X we denote the feasible set of the SIP, i.e.

$$X = \{x \in \Re^n : \ g(x,y) \le 0, \ \forall\, y \in \mathcal{Y}\}$$

or equivalently,

$$X = \bigcap_{y \in \mathcal{Y}} \left\{x \in \Re^n : \ g(x,y) \le 0\right\}.$$

Since $g(\cdot, y)$ is continuous w.r.t. x, X is the possibly infinite intersection of closed sets, and hence a closed set itself. In addition, let us denote by X^0, $\mathcal{Y}^0$, $\partial X = X \setminus X^0$ and $\partial\mathcal{Y} = \mathcal{Y} \setminus \mathcal{Y}^0$ the topological interior, boundary of X and $\mathcal{Y}$, respectively. For each $y \in \mathcal{Y}$, we denote the set of active indices at y by

$$I_h(y) = \{\ell \in L \ : \ h_\ell(y) = 0\}$$

and if $\bar{x} \in X$, we denote by $X_{\text{act}}(\bar{x})$ the set of corresponding active points:

$$X_{\text{act}}(\bar{x}) = \{y \in \mathcal{Y} : \ g(\bar{x}, y) = 0\}.$$

These active points are in fact (global) maxima for $g(\bar{x}, \cdot)$ on $\mathcal{Y}$, which play an important role for the local structure of X at $\bar{x}$ and hence, also for local optimality conditions. Moreover, there is the striking difference from $|\mathcal{Y}| < \infty$. unless $\mathcal{Y}$ is a finite set, the set $X_{\text{act}}(x)$ need not be a subset of $X_{\text{act}}(\bar{x})$ for $X \ni x$ arbitrarily close to $\bar{x}$. In particular, the active set $X_{\text{act}}(x)$ changes from point to point along the boundary of the feasible set X.

In what follows, we begin with a description of necessary conditions for the SIP which we will use in the sequel. For this, we adopt a commonly used way to transform SIP into a problem of bilevel type, and then to reduce the latter to a standard NLP. Under suitable assumptions, we wish to derive first-order optimality conditions for the SIP (see [12] for related topic). First let us consider the following n-parametric optimization problem

$$\min_y \ -g(x,y) \quad \text{s. t.} \ y \in \mathcal{Y}, \tag{2.2}$$

which is, also called the lower level problem, well defined in the sense that problem (2.2) always has solutions. Clearly, it is $x \in X$ if and only if a solution y of (2.2) satisfies $g(x, y) \leq 0$. Thus, the condition $x \in X$ is equivalent with $x \in \mathrm{Proj}_x(\widetilde{X})$, where

$$\widetilde{X} = \{(x, y) \in \Re^n \times \Re^m : \ g(x, y) \leq 0 \text{ and } y \text{ is a solution of (2.2)}\},$$

here, $\mathrm{Proj}_x(\cdot)$ denotes the orthogonal projection of $\tilde{X}$ onto the space $\Re^n$. Summarizing, the bilevel (BL) formulation of SIP is given by

$$\begin{array}{ll} \min & f(x) \\ \text{s. t.} & g(x, y) \leq 0, \text{ and } y \text{ is a solution of} \\ & \min\limits_{y} \ -g(x, y) \quad \text{s. t. } y \in \mathcal{Y}. \end{array} \tag{2.3}$$

In this form, the SIP can be considered as a special instance of a BL-problem. In principle, by using BL-case's techniques, optimality conditions for the SIP can be directly deduced from the corresponding results in BL programming (BLP). But, sometimes optimality conditions obtained in this way are so complex that it is very difficult to apply a homotopy method to this system. Hence, we try to illustrate first-order necessary conditions for the original problem (2.1) by applying the so-called reduction approach in [6]. For this, it is necessary to make the following assumptions hold, which will be used throughout this paper.

(A1) For each fixed $x \in \Re^n$, the function $g(x, \cdot)$ is uniformly strictly concave w.r.t. the variable y. All functions $h_\ell(y)$, $\ell \in L$, are convex w.r.t. the variable y.
(A2) The sets X, $\mathcal{Y}$ are bounded respectively, and $X^0 \neq \emptyset$, $\mathcal{Y}^0 \neq \emptyset$ (Slater's condition).
(A3) For any $x \in X$ with $X_{\mathrm{act}}(x) \neq \emptyset$, the vectors $\nabla_x g(x, y), \forall y \in X_{\mathrm{act}}(x)$, are linearly independent (LICQ in SIP).
(A4) For any $y \in \mathcal{Y}$, $\{ \nabla h_\ell(y) : \ell \in I_h(y)\}$ is of full column rank (LICQ in NLP).

It is well-known that under assumption (A1), for any fixed $x \in \Re^n$, there exists a unique (global) minimizer for problem (2.2), defined as $y(x)$. Furthermore, under assumption (A2), $y := y(x)$ is a solution to (2.2) if and only if there exists $\lambda \in \Re^l$ such that (y, λ) satisfies the following first-order necessary conditions for (2.2), given by

$$\begin{array}{l} \nabla_y \mathcal{L}(x, y, \lambda) = 0, \\ \lambda^{\mathrm{T}} h(y) = 0, \\ \lambda \geq 0, \quad h(y) \leq 0, \end{array} \tag{2.4}$$

where $h = (h_1, \ldots, h_l)^{\mathrm{T}}$ for brevity and for every parameter $x \in \Re^n$, the Lagrange function of (2.2) at $y \in \Re^m$ is denoted by

$$\mathcal{L}(x, y, \lambda) = -g(x, y) + \sum_{\ell=1}^{l} \lambda_\ell h_\ell(y).$$

In this case the semi-infinite constraint can be replaced locally by the finite (but in general nonlinear) constraint $G(x) = g(x, y(x))$. Thus, the SIP (2.1) or (2.3) is certainly locally equivalent to the so-called reduced problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s. t.} & G(x) := g(x, y(x)) \leq 0. \end{array} \tag{2.5}$$

Note that (2.5) is a finite optimization problem but not necessary to be smooth. If some CQs hold, standard optimality conditions can be applied to (2.5), and then lead to that for the SIP problem. Moreover, for algorithmic purpose, the smoothness of $G(\cdot)$ must be required. To this end, we have to consider the analytic properties of $G(x)$. Hence, it is necessary to list some results from sensitivity analysis in NLP [8].

Lemma 2.1 *For any fixed $\bar{x} \in \Re^n$, let $\bar{y}$ be a solution of* (2.2)*, and the following conditions* (1)−(3) *hold at $\bar{y}$ for* (2.2)*:*

(1) *there exists Lagrange multiplier vector $\bar{\lambda} \in \Re^l$ such that, $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies* (2.4).
(2) *Assumption* (A4) *holds.*
(3) *the strong second-order sufficient optimality condition:*

$$\eta^{\mathrm{T}} \nabla_y^2 \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})\eta > 0 \quad \text{for all } \Re^m \ni \eta \neq 0 \quad s.t. \nabla h_\ell(\bar{y})\eta = 0, \quad \text{if } \lambda_\ell > 0, \ell \in L. \text{ (SSOSC)}$$

Then there exist a neighborhood $\Theta(\bar{x})$ of $\bar{x}$ and unique Lipschitz continuous functions $y(\cdot)$, $\lambda(\cdot)$ on $\Theta(\bar{x})$, such that $y(\bar{x}) = \bar{y}$, $\lambda(\bar{x}) = \bar{\lambda}$, and for $x \in \Theta(\bar{x})$:

(a) *$y(x)$ with Lagrange multiplier vector $\lambda(x)$ satisfy the same conditions* (1)−(3) *and $y(x)$ is an isolated strict local minimum of* (2.2).
(b) *the following relation holds: $\bar{J}^+ \subset J(x) \subset \bar{J}$, where*

$$J(x) = \{\ell \in L : \ h_\ell(y(x)) = 0\}, \quad \bar{J} = J(\bar{x}), \quad \bar{J}^+ = \{j \in \bar{J} : \ \bar{\lambda}_\ell > 0\}.$$

(c) *the local optimal value function $G(x) := g(x, y(x))$ is once continuously differentiable, i.e., of class C^1, and twice directionally differentiable.*
(d) *if at $\bar{y}$ the strict complementarity condition is satisfied, i.e., $\bar{\lambda}_\ell > 0$ for all $\ell \in \bar{J}$, then the function $G(x)$ are C^2-function in $\Theta(\bar{x})$ with derivative*

$$\begin{aligned} \nabla G(x) &= \nabla_x g(x, y(x)) + \nabla_y g(x, y(x))\dot{y}(x) \\ &= \nabla_x g(x, y(x)) \end{aligned}$$

and $\bar{J} = J(x) = \bar{J}^+$.

Definition 2.1 We say that, the linear independence constraint qualification (LICQ) in NLP holds at $\bar{x} \in X$, if for the constraint in (2.5), we have

$$\nabla G(\bar{x}) \neq 0,$$

or equivalently,

$$\nabla_x g(\bar{x}, \bar{y}) \neq 0 \quad \text{for any } \bar{y} \in X_{\mathrm{act}}(\bar{x}).$$

The above arguments show that, at least locally near $\bar{x}$, if $\nabla G(\bar{x}) \neq 0$, the standard optimality conditions of finite optimization can be applied to (2.5), and then lead to obtain optimality conditions for the semi-infinite problem: There exists some multiplier vector $u \in \Re_+$ such that a pair (x, u) satisfies

$$\begin{aligned} &\nabla f(x) + \nabla G(x)u = 0, \\ &u^{\mathrm{T}} G(x) = 0, \\ &u \geq 0, \ G(x) \leq 0, \end{aligned} \tag{2.6}$$

where $G(x) = g(x, y(x))$, here, $y(x)$ satisfies implicitly the following KKT-conditions for (2.2).

$$\begin{aligned} &-\nabla_y g(x, y) + \nabla h(y)v = 0, \\ &v^{\mathrm{T}} h(y) = 0, \\ &v \geq 0, \ h(y) \leq 0. \end{aligned} \tag{2.7}$$

We call $x \in \Re^n$ with $y \in \Re^m$, $u \in \Re$ and $v \in \Re^l$, satisfying (2.6) associated with (2.7), a stationary point of the SIP problem. In fact, for more general problems of this type, one can obtain the so-called substationary point (see [15] for more details).

Note that (2.6), together with (2.7), is a system of $n + m + 1 + l$ nonlinear equations of $n + m + 1 + l$ unknowns. If given a sufficiently accurate starting point, these equations can then be solved very fast by Newton's method. However, a good initial point is rarely available. So, the special techniques to generate such a point have to be applied. In this paper we apply a successful homotopy method for NLP to the SIP problem.

3 Homotopy algorithm and its global convergence

In this section, we propose a globally convergent homotopy method for solving (2.1) on the basis of (2.6) and (2.7). To ensure that the global convergent theory for homotopy algorithm holds in terms of conditions on the objective function and constraints, we further assume that the following additional condition (A5) is true.

(A5) For any $x \in \partial X$ with $X_{\mathrm{act}}(x) \neq \emptyset$, the following condition holds

$$\{\, x + \nabla_x g(x, y(x))u \;:\; u > 0 \,\} \bigcap X = \{x\},$$

where $y(x) \in X_{\mathrm{act}}(x)$.

To solve (2.6), together with (2.7), we construct the following homotopy function $H : \Re^{n+m+1+l} \times (0, 1] \to \Re^{n+m+1+l}$, defined by

$$H(w, w^{(0)}, \mu) \equiv \begin{pmatrix} (1-\mu)(\nabla f(x) + \nabla_x g(x, y)u) + \mu(x - x^{(0)}) \\ (1-\mu)(-\nabla_y g(x, y) + \nabla h(y)v) + \mu(y - y^{(0)}) \\ ug(x, y) - \mu u^{(0)} g(x^{(0)}, y^{(0)}) \\ Vh(y) - \mu V^{(0)} h(y^{(0)}) \end{pmatrix} = 0, \qquad (3.1)$$

where

$$w = (x, y, u, v), w^{(0)} = (x^{(0)}, y^{(0)}, u^{(0)}, v^{(0)}) \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$$

and

$$V = \mathrm{diag}(v), \quad V^{(0)} = \mathrm{diag}(v^{(0)}),$$

$$\Theta^0 = \{\, (x, y) \in X \times \mathcal{Y} :\; g(x, y) < 0, h(y) < 0 \,\}, \Theta = \mathrm{cl}\{\Theta^0\}, \partial\Theta = \Theta \backslash \Theta^0.$$

In addition, we rewrite $H(w, w^{(0)}, \mu)$ as $H_{w^{(0)}}(w, \mu)$ for brevity.

Eq. (3.1) is called a *combined homotopy* and the corresponding algorithm as the *combined homotopy interior point method* because the first and second components of (3.1) are the linear homotopy, while the third and fourth components of it, which make the method an interior point method, are the Newton homotopy.

Eq. (3.1) contains two limiting problems. On the one hand when $\mu = 1$, we have

$$\begin{aligned} x - x^{(0)} &= 0, \\ y - y^{(0)} &= 0, \\ ug(x, y) - u^{(0)} g(x^{(0)}, y^{(0)}) &= 0, \\ Vh(y) - V^{(0)} h(y^{(0)}) &= 0, \end{aligned} \qquad (3.2)$$

which has an unique solution $w = w^{(0)}$. On the other hand when $\mu = 0$, we have

$$H_{w^{(0)}}(w, 0) = \begin{pmatrix} \nabla f(x) + \nabla_x g(x, y)u \\ -\nabla_y g(x, y) + \nabla h(y)v \\ ug(x, y) \\ Vh(y) \end{pmatrix} = 0, \tag{3.3}$$

which is the problem we want to solve. For a given $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, the zero point set of $H_{w^{(0)}}(\cdot, \cdot)$ in $\Theta^0 \times \Re_{++} \times \Re^l_{++} \times (0, 1]$ is defined as:

$$H^{-1}_{w^{(0)}}(0) = \left\{(w, \mu) \in \Theta^0 \times \Re_{++} \times \Re^l_{++} \times (0, 1] : H_{w^{(0)}}(w, \mu) = 0\right\}.$$

The method sketched so far is based on the assumption that there exists a smooth solution path without bifurcation points, which starts from $(w^{(0)}, 1)$ and approaches to the hyperplane at $\mu = 0$. Before describing homotopy continuation methods in more algorithmic details we look for criteria for an existence of such a solution path.

Parameterized Sard theorem [2] Let Q, N and P be smooth manifolds of dimensions q, m and p, respectively. Let $\Phi : Q \times N \to P$ be a C^r map, where $r > \max\{0, m - p\}$. If $0 \in P$ is a regular value of Φ, then for almost all $a \in Q$, 0 is a regular value of $\Phi_a \equiv \Phi(a, \cdot)$.

Lemma 3.1 *Suppose that Assumption* (A2) *holds, and let* $H_{w^{(0)}}(w, \mu)$ *be defined in* (3.1)*, then for almost all* $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, 0 *is a regular value of* $H_{w^{(0)}}(\cdot, \cdot)$*, and* $H^{-1}_{w^{(0)}}(0)$ *consists of some smooth curves, among them, a smooth curve, say* $\Gamma_{w^{(0)}}$*, is starting from* $(w^{(0)}, 1)$.

Proof By $\mathcal{D}H(w, w^{(0)}, \mu)$ we denote the Jacobi matrix of $H(w, w^{(0)}, \mu)$ w.r.t. all variables w, $w^{(0)}$, μ, given by

$$\mathcal{D}H(w, w^{(0)}, \mu) = \left(\frac{\partial H(w, w^{(0)}, \mu)}{\partial w}, \frac{\partial H(w, w^{(0)}, \mu)}{\partial w^{(0)}}, \frac{\partial H(w, w^{(0)}, \mu)}{\partial \mu}\right).$$

For any $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$ and $\mu \in (0, 1]$, we obtain

$$\frac{\partial H(w, w^{(0)}, \mu)}{\partial w^{(0)}} = \begin{pmatrix} -\mu I & 0 & 0 & 0 \\ 0 & -\mu I & 0 & 0 \\ -\mu u^{(0)} g'_{x^{(0)}}(x^{(0)}, y^{(0)}) & -\mu u^{(0)} g'_{y^{(0)}}(x^{(0)}, y^{(0)}) & -\mu g(x^{(0)}, y^{(0)}) & 0 \\ 0 & -\mu V^{(0)} \nabla h(y^{(0)}) & 0 & -\mu \mathrm{diag}(h(y^{(0)})) \end{pmatrix}. \tag{3.4}$$

By a straightforward calculation, the determinant of (3.4) is

$$\left|\frac{\partial H(w, w^{(0)}, \mu)}{\partial w^{(0)}}\right| = (-\mu)^{n+m+1+l} g(x^{(0)}, y^{(0)}) \prod_{\ell=1}^{l} h_\ell(y^{(0)}).$$

From $(x^{(0)}, y^{(0)}) \in \Theta^0$, we have $g(x^{(0)}, y^{(0)}) < 0$, $h_\ell(y^{(0)}) < 0$, and hence

$$|\partial H(w, w^{(0)}, \mu)/\partial w^{(0)}| \neq 0,$$

which implies that $\mathcal{D}H(w, w^{(0)}, \mu)$ is of full row rank. That is, 0 is a regular value of $H(w, w^{(0)}, \mu)$. By the parameterized Sard theorem, for almost all $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, 0 is a regular value of $H_{w^{(0)}} : \Theta^0 \times \Re_{++} \times \Re^l_{++} \times (0, 1] \to \Re^{n+m+1+l}$. By the implicit function theorem, $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Again, because $H_{w^{(0)}}(w^{(0)}, 1) = 0$, there must be a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)}, 1)$. □

Lemma 3.2 *Suppose that assumptions* $(A1) - (A5)$ *hold, for a given* $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, *if* 0 *is a regular value of* $H_{w^{(0)}}(\cdot, \cdot)$, *then* $\Gamma_{w^{(0)}}$ *is a bounded curve in* $\Theta \times \Re_+ \times \Re^l_+ \times [0, 1]$.

Proof If not then, there exists a sequence of points $\{(w^{(k)}, \mu_k)\} \subset \Gamma_{w^0}$ such that $||(w^{(k)}, \mu_k)|| \to \infty$. Because X, $\mathcal{Y}$ and $[0, 1]$ are bounded, there exists a subsequence of points, denoted also by $\{(w^{(k)}, \mu_k)\}$, such that $x^{(k)} \to x^* \in X$, $y^{(k)} \to y^* \in \mathcal{Y}$, $\mu_k \to \mu^* \in [0, 1]$ and $||(u^{(k)}, v^{(k)})|| \to \infty$, as $k \to \infty$. Thus, we can set

$$I(y^*) = \{j \in \{1, \ldots, l\} : v^{(k)}_j \to +\infty\}.$$

So, at least one of two cases (a) and (b):

$$\text{(a) } I(y^*) \neq \emptyset, \qquad \text{(b) } I(y^*) = \emptyset$$

hold. Next, we prove that each of two cases a) and b) can not occur by considering the following three cases: $\mu^* = 1$, $\mu^* \in (0, 1)$ and $\mu^* = 0$.

(1) When $\mu^* = 1$.

From the fourth equality of (3.1), it follows that

$$h_j(y^{(k)}) = \mu_k (v^{(k)}_j)^{-1} v^{(0)}_j h_j(y^{(0)}), \quad j = 1, \ldots, l, \tag{3.5}$$

where $v^{(k)}_j$ denotes the jth element of $v^{(k)}$.

If the (a) appears, i.e., $I(y^*) \neq \emptyset$.

It is easy to see that the binding set $I(y^*)$ coincides with the active index set $I_h(y^*)$, i.e., $I(y^*) = I_h(y^*)$, and then $y^* \in \partial\mathcal{Y}$.

By the second equality of (3.1), we have

$$(1 - \mu_k)(-\nabla_y g(x^{(k)}, y^{(k)}) + \nabla h(y^{(k)}) v^{(k)}) + \mu_k (y^{(k)} - y^{(0)}) = 0. \tag{3.6}$$

Setting $k \to +\infty$, then (3.6) reduces to

$$\sum_{j \in I_h(y^*)} \lim_{k \to +\infty} [(1 - \mu_k) v^{(k)}_j] \nabla h_j(y^*) + y^* = y^{(0)},$$

which contradicts with that, the outer normal cone of $\mathcal{Y}$ meets itself at its boundary point y^*. Hence, case a) is impossible.

If the (b) appears, i.e., $I(y^*) = \emptyset$, which implies that $u^{(k)} \to +\infty$.

From the third equality of (3.1), it follows that

$$g(x^{(k)}, y^{(k)}) = \mu_k (u^{(k)})^{-1} u^{(0)} g(x^{(0)}, y^{(0)}). \tag{3.7}$$

Setting $k \to +\infty$, we obtain

$$g(x^*, y^*) = 0.$$

This implies $y^* \in X_{\text{act}}(x^*) \neq \emptyset$ because of $y^* \in \mathcal{Y}$. From (A3) we have $\nabla_x g(x^*, y^*) \neq 0$.

By the second equality of (3.1), we have

$$(1-\mu_k)(\nabla f(x^{(k)})+\nabla_x g(x^{(k)},y^{(k)})u^{(k)})+\mu_k(x^{(k)}-x^{(0)})=0. \tag{3.8}$$

In the following, we will prove that Eq. (3.8) and $u^{(k)}\to+\infty$ will yield a contradiction and hence case b) is impossible.

Setting $k\to+\infty$, from (3.8) and $\nabla_x g(x^*,y^*)\neq 0$ it is easy to see that the limit of $(1-\mu_k)u^{(k)}$, denoted by $\bar{\alpha}\geq 0$, exists, and we have

$$x^{(0)}=x^*+\bar{\alpha}\nabla_x g(x^*,y^*). \tag{3.9}$$

If $\bar{\alpha}=0$, then from $X_{\rm act}(x^*)\neq\emptyset$ it follows that $x^*\in\partial X$, which contradicts with $x^{(0)}\in X^0$, else (3.9) contradicts with the condition (A5). Hence, case (b) is impossible.

(2) When $\mu^*\in(0,1)$.

If the a) appears, from (3.5) we also know that the binding set $I(y^*)$ coincides with the active index set $I_h(y^*)$, i.e., $I(y^*)=I_h(y^*)$, and then $y^*\in\partial\mathcal{Y}$. From (A4) it follows that $\nabla h_j(y^*)\neq 0$ for all $j\in I_h(y^*)$.

By the second equality of (3.1), we have

$$\lim_{k\to+\infty}\Big((1-\mu_k)(-\nabla_y g(x^{(k)},y^{(k)})+\nabla h(y^{(k)})v^{(k)})+\mu_k(y^{(k)}-y^{(0)})\Big)=0. \tag{3.10}$$

Using $x^{(k)}\to x^*$, $y^{(k)}\to y^*$ $(k\to+\infty)$, (3.12) becomes

$$\sum_{j\in I_h(y^*)}(1-\mu^*)\nabla h_j(y^*)\lim_{k\to+\infty}(v_j^{(k)})=-\mu^*(y^*-y^{(0)})-\sum_{j\notin I_h(y^*)}\lim_{k\to+\infty}[(1-\mu_k)v_j^{(k)}]\nabla h_j(y^*),$$

in which the left-hand side tends to infinity, but the right-hand side is bounded. Hence, case a) is impossible.

If the (b) appears, that is, $u^{(k)}\to+\infty$. By (3.7) and setting $k\to+\infty$, we also obtain that

$$g(x^*,y^*)=0,$$

which implies that $y^*\in X_{\rm act}(x^*)$ because of $y^*\in\mathcal{Y}$. From (A3) we have $\nabla_x g(x^*,y^*)\neq 0$.

By (3.8) and setting $k\to+\infty$, we have

$$(1-\mu^*)\nabla_x g(x^*,y^*)\lim_{k\to+\infty}u^{(k)}=-\mu^*(x^*-x^{(0)})+(1-\mu^*)\nabla f(x^*). \tag{3.11}$$

The term in the left-hand side of (3.13) tends to infinity, but the right-hand side is bounded. Hence, case (b) is impossible.

(3) When $\mu^*=0$.

If the (a) appears, from (3.5) we also know that $I(y^*)=I_h(y^*)\neq\emptyset$ and $y^*\in\partial\mathcal{Y}$. From (A4) it follows that $\nabla h_j(y^*)\neq 0$ for all $j\in I_h(y^*)$.

By the second equality of (3.1), we have

$$\lim_{k\to+\infty}\Big((1-\mu_k)(-\nabla_y g(x^{(k)},y^{(k)})+\nabla h(y^{(k)})v^{(k)})+\mu_k(y^{(k)}-y^{(0)})\Big)=0. \tag{3.12}$$

Using $x^{(k)}\to x^*$, $y^{(k)}\to y^*$ $(k\to+\infty)$, (3.12) becomes

$$\sum_{j\in I_h(y^*)}\nabla h_j(y^*)\lim_{k\to+\infty}v_j^{(k)}=-\sum_{j\notin I_h(y^*)}\nabla h_j(y^*)\lim_{k\to+\infty}v_j^{(k)}$$

in which the left-hand side tends to infinity, but the right-hand side is bounded. Hence, case a) is impossible.

If the (b) appears, we have $u^{(k)} \to +\infty$. From the discussion in Cases (1) and (2) it is not hard to see that $\nabla_x g(x^*, y^*) \neq 0$. By (3.8) and setting $k \to +\infty$, we have

$$\nabla_x g(x^*, y^*) \lim_{k \to +\infty} u^{(k)} = \nabla f(x^*). \tag{3.13}$$

The term in the left-hand side of (3.13) tends to infinity, but the right-hand side is bounded. Hence, Case (b) is impossible.

Finally, from (1), (2) and (3) we know that the Γ_{w^0} is bounded. □

The following theorem which implies the global convergence of interior path-following method for the SIP, can be proven.

Theorem 3.1 *Suppose that Assumptions (A1)–(A5) hold, let $H_{w^{(0)}}(w, \mu)$ be defined in* (3.1)*, then the system* (2.6)*, with* (2.7)*, have at least one solution. For almost all $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, the zero-point set $H^{-1}_{w^{(0)}}(0)$ of* (3.1) *contains a smooth curve $\Gamma_{w^{(0)}} \subset \Theta^0 \times \Re_{++} \times \Re^l_{++} \times (0, 1]$, which starts from $(w^{(0)}, 1)$ and tends to the hyperplane at $\mu = 0$. As $\mu \to 0^+$, the limit set $\Gamma \times \{0\} \subset \Theta \times \Re_+ \times \Re^l_+ \times [0, 1]$ of $\Gamma_{w^{(0)}}$ is nonempty, and every point in Γ is the solution of* (2.6) *together with* (2.7)*.*

Proof By Lemma 3.1, for almost all $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, 0 is a regular value of $H_{w^{(0)}}$, and $\Gamma_{w^{(0)}}$ consists of some smooth curves, among them, a smooth curve is starting from $(w^{(0)}, 1)$.

By the classification theorem of one-dimensional smooth manifold, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or a unit interval $(0, 1]$. Note that the matrix is of the form

$$\begin{aligned}\nabla H_{w^{(0)}}(w^{(0)}, 1) &= \left.\frac{\partial H_{w^{(0)}}(w, \mu)}{\partial w}\right|_{(w=w^{(0)},\ \mu=1)} \\ &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ u^{(0)} \nabla_x g(x^{(0)}, y^{(0)})^{\mathrm{T}} & u^{(0)} \nabla_y g(x^{(0)}, y^{(0)})^{\mathrm{T}} & g(x^{(0)}, y^{(0)}) & 0 \\ 0 & V^{(0)} \nabla h(y^{(0)})^{\mathrm{T}} & 0 & h(y^{(0)}) \end{pmatrix}\end{aligned}$$

and that $g(x^{(0)}, y^{(0)}) < 0$ and $h(y^{(0)}) < 0$, we see easily that $\partial H_{w^{(0)}}(w^{(0)}, 1)/\partial w$ is non-singular, hence $\Gamma_{w^{(0)}}$ is diffeomorphic to $(0, 1]$. Let (w^*, μ^*) be a limit point of $\Gamma_{w^{(0)}}$, then only the following three cases are possible.

(1) $(w^*, \mu^*) \in \left(\Theta \times \Re_+ \times \Re^l_{++}\right) \times \{1\}$;

(2) $(w^*, \mu^*) \in \partial\left(\Theta \times \Re_+ \times \Re^l_{++}\right) \times (0, 1]$;

(3) $(w^*, \mu^*) \in \left(\mathrm{cl}(\Theta) \times \Re_+ \times \Re^l_+\right) \times \{0\}$.

Note that the equation $H(w, w^{(0)}, 1) = 0$ has only one solution $(w^{(0)}, 1)$ in $\Theta^0 \times \Re_{++} \times \Re^l_{++} \times \{1\}$, and hence the case (1) is impossible. In case (2), there must exist a sequence of $(w^{(k)}, \mu_k) \in \Gamma_{w^{(0)}}$ such that, $h_j(y^{(k)}) \to 0$ for some $1 \leq j \leq l$ (or $g(x^{(k)}, y^{(k)}) \to 0$). Then, from the fourth (third) equation of (3.1), it follows that $||v_j^{(k)}|| \to \infty$ (or $u_k \to +\infty$). This contradicts with lemma 3.2. Thus, the case (2) is impossible. As a conclusion, the case (3) is the only possible case, i.e., $\Gamma_{w^{(0)}}$ must approach to the hyperplane at $\mu = 0$. In this case, from the boundedness of $(w^{(k)}, \mu_k)$ on the curve

$\Gamma_{w^{(0)}}$ and $\mu_k \in (0,1]$, we know that $\{w^{(k)}\}$ have at least one accumulation point as $\mu_k \to 0^+$. Let x^*, y^*, u^* and v^* the accumulation points of $\{x^{(k)}\}$, $\{y^{(k)}\}$, $\{u^{(k)}\}$ and $\{v^{(k)}\}$, respectively. By (3.1), we have

$$\begin{aligned}(1-\mu_k)(\nabla f(x^{(k)}) + \nabla_x g(x^{(k)}, y^{(k)})u^{(k)}) + \mu_k(x^{(k)} - x^{(0)}) = 0,\\ (1-\mu_k)(-\nabla_y g(x^{(k)}, y^{(k)}) + \nabla h(y^{(k)})v^{(k)}) + \mu_k(y^{(k)} - y^{(0)}) = 0,\\ u^{(k)} g(x^{(k)}, y^{(k)}) - \mu_k u^{(0)} g(x^{(0)}, y^{(0)}) = 0,\\ V^{(k)} h(y^{(k)}) - \mu_k V^{(0)} h(y^{(0)}) = 0.\end{aligned} \tag{3.14}$$

In the following, we will prove that x^* and y^*, with the multipliers u^* and v^*, are indeed the solution of (2.6) with (2.7). By the continuity of H, and passing to the limit in the first and second equation of (3.14), we obtain

$$\begin{aligned}\nabla f(x^*) + \nabla_x g(x^*, y^*)u^* = 0,\\ -\nabla_y g(x^*, y^*) + \nabla h(y^*)v^* = 0.\end{aligned}$$

By the third equation of (3.14), i.e.,

$$u^{(k)} g(x^{(k)}, y^{(k)}) = \mu_k u^{(0)} g(x^{(0)}, y^{(0)}) \tag{3.15}$$

and $u^{(0)} > 0$ and $g(x^{(0)}, y^{(0)}) < 0$, we have $u^{(k)} \neq 0$, $g(x^{(k)}, y^{(k)}) \neq 0$ for all $\mu_k \neq 0$. So, we have $u^{(k)} > 0$ and $g(x^{(k)}, y^{(k)}) < 0$ for all $\mu_k \neq 0$. By the continuity of $g(x, y)$, passing to the limit in (3.15), we obtain

$$\begin{aligned}u^* g(x^*, y^*) = 0,\\ u^* \geq 0,\ g(x^*, y^*) \leq 0.\end{aligned}$$

By the fourth equation of (3.14), i.e.,

$$v_i^{(k)} h_i(y^{(k)}) = \mu_k v_i^{(0)} h_i(y^{(0)}), \quad i = 1, \ldots, l \tag{3.16}$$

and $v_i^{(0)} > 0$ and $h_i(y^{(0)}) < 0$, we have $v_i^{(k)} \neq 0$ and $h_i(y^{(k)}) \neq 0$ for all $\mu_k \neq 0$. Hence, we obtain that $v_i^{(k)} > 0$ and $h_i(y^{(k)}) < 0$ for all $\mu_k \neq 0$. By the continuity of $h_i(y)$, passing to the limit in (3.16), we obtain

$$\begin{aligned}v_i^* h_i(y^*) = 0, \quad i = 1, \ldots, l,\\ v_i^* \geq 0,\ h_i(y^*) \leq 0.\end{aligned}$$

As a conclusion, x^* and y^*, with the multipliers u^* and v^*, are the solution of (2.6) with (2.7). □

From theorem 3.1, for almost all $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re_{++}^l$, the homotopy equations in (3.1) implicitly define a smooth curve $\Gamma_{w^{(0)}}$, called the *homotopy path*. Let s denote the arclength of $\Gamma_{w^{(0)}}$, we can parameterize $\Gamma_{w^{(0)}}$ with respect to s, that is, there exist continuously differentiable functions $w(s)$, $\mu(s)$ such that

$$\begin{aligned}H_{w^{(0)}}(w(s), \mu(s)) = 0,\\ ||(\dot{w}(s), \dot{\mu}(s))|| = 1,\\ w(0) = w^{(0)}, \mu(0) = 1,\\ \dot{\mu}(0) < 0.\end{aligned} \tag{3.17}$$

By differentiating the first equation of (3.17) we obtain the following result.

Theorem 3.2 *The homotopy path $\Gamma_{w^{(0)}}$ is determined by the following initial value problem to the system of ordinary differential equations*

$$\begin{aligned} \nabla H_{w^{(0)}}(w,\mu)\begin{pmatrix}\dot{w}(s)\\ \dot{\mu}(s)\end{pmatrix} &= 0,\\ ||(\dot{w}(s),\dot{\mu}(s))|| &= 1,\\ w(0) = w^{(0)}, \mu(0) &= 1,\\ \dot{\mu}(0) &< 0. \end{aligned} \tag{3.18}$$

And the w-component of $(w(s^*),\mu(s^*))$, for $\mu(s^*) = 0$, is the solution of (2.6) *with* (2.7).

Based on theorems 3.1, 3.2 and by using (3.17) and (3.18), we describe how to numerically trace the homotopy path $\Gamma_{w^{(0)}}$ by some predictor-corrector (PC) procedure [1]. The idea in PC methods is to numerically trace the curve $\Gamma_{w^{(0)}}$ by generating a sequence of points $\{z^{(k)}\} \subset \Theta^0 \times \Re_{++} \times \Re^l_{++} \times (0,1]$ with $z^{(0)} = (w^{(0)},1)$, that lie approximately on the curve in order of increasing arc length, i.e. $z^{(k)} \approx (w(s_k),\mu(s_k))$, where $\{s_k\}$ is some increasing sequence of arc length. To obtain a new point $z^{(k+1)}$ along the curve $\Gamma_{w^{(0)}}$, we first make a predictor step. The prediction phase requires for each iterate $z^{(k)}$ the corresponding unit tangent vector to the curve $(z')^{(k)} \approx (\dot{w}(s_k),\dot{\mu}(s_k))$. Recall that for almost all $w^{(0)} \in \Theta^0 \times \Re_{++} \times \Re^l_{++}$, the Jacobian matrix $\nabla H_{w^{(0)}}(w,\mu)$ w.r.t. w and μ has full row rank. Thus, this is accomplished by finding an element η in the kernel of $\nabla H_{w^{(0)}}(w,\mu)$ that maintains the correct orientation, viz.,

$$\left(\nabla_w H_{w^{(0)}}(w,\mu), \nabla_\mu H_{w^{(0)}}(w,\mu)\right)\begin{pmatrix}\dot{w}(s)\\ \dot{\mu}(s)\end{pmatrix} = 0,$$

where

$$\eta = \begin{pmatrix}\dot{w}(s)\\ \dot{\mu}(s)\end{pmatrix}$$

and setting $(z')^{(k)} = \pm\frac{\eta}{||\eta||}$. Since the negative direction will lead us back to the initial point, so we must go along the positive direction. The criterion in step 1(b) of Algorithm 3.1 that determines the positive direction is based on a basic theory of homotopy method, namely, the positive direction η at any point (w,μ) on $\Gamma_{w^{(0)}}$ keeps the sign of the determinant

$$\begin{vmatrix}\nabla H_{w^{(0)}}(w,\mu)\\ \eta^{\mathrm{T}}\end{vmatrix}$$

invariant. On the first iterate, the sign is determined by the following lemma.

Lemma 3.3 *If $\Gamma_{w^{(0)}}$ is smooth, then the positive direction $\eta^{(0)}$ at the initial point $(w^{(0)},\mu_0)$ satisfies*

$$sign\begin{vmatrix}\nabla H_{w^{(0)}}(w^{(0)},1)\\ {\eta^{(0)}}^{\mathrm{T}}\end{vmatrix} = (-1)^{l+2}. \tag{3.19}$$

Proof From the definition of $\nabla H_{w^{(0)}}(w,\mu)$, it follows that

$$\begin{aligned}\nabla H_{w^{(0)}}(w,\mu) &= \frac{\partial H_{w^{(0)}}(w,\mu)}{\partial(w,\mu)}\\ &= \begin{pmatrix} Q_1 & (1-\mu)u\nabla^2_{yx}g(x,y) & (1-\mu)\nabla_x g(x,y) & 0 & a\\ -(1-\mu)\nabla^2_{xy}g(x,y) & Q_2 & 0 & (1-\mu)\nabla h(y) & b\\ u\nabla_x g(x,y)^{\mathrm{T}} & u\nabla_y g(x,y)^{\mathrm{T}} & g(x,y) & 0 & c\\ 0 & V\nabla h(y)^{\mathrm{T}} & 0 & h(y) & d\end{pmatrix},\end{aligned}$$

where

$$
\begin{aligned}
&Q_1 = (1-\mu)\left(\nabla^2 f(x) + u\nabla_x^2 g(x,y)\right) + \mu I \in \Re^{n\times n},\\
&Q_2 = (1-\mu)\left(-\nabla_y^2 g(x,y) + \nabla^2 h(y)v\right) + \mu I \in \Re^{m\times m},\\
&a = x - x^{(0)} - \nabla f(x) - \nabla_x g(x,y)u,\\
&b = y - y^{(0)} + \nabla_y g(x,y) - \nabla h(y)v,\\
&c = -u^{(0)} g(x^{(0)}, y^{(0)}),\\
&d = -V^{(0)} h(y^{(0)}).
\end{aligned}
$$

Using the initial point $w^{(0)} \in \Theta^0$ and $\mu_0 = 1$, we obtain

$$
\begin{aligned}
&\nabla H_{w^{(0)}}(w^{(0)}, 1)\\
&= \begin{pmatrix}
I & 0 & 0 & 0 & a^{(0)}\\
0 & I & 0 & 0 & b^{(0)}\\
u^{(0)}\nabla_x g(x^{(0)}, y^{(0)})^{\mathrm{T}} & u^{(0)}\nabla_y g(x^{(0)}, y^{(0)})^{\mathrm{T}} & g(x^{(0)}, y^{(0)}) & 0 & c^{(0)}\\
0 & V^{(0)}\nabla h(y^{(0)})^{\mathrm{T}} & 0 & h(y^{(0)}) & d^{(0)}
\end{pmatrix},\\
&= (M_1, M_2),
\end{aligned}
$$

where $M_1 \in \Re^{(n+m+1+l)\times(n+m+1+l)}$, $M_2 \in \Re^{(n+m+1+l)\times 1}$. The tangent vector $\eta^{(0)} = (\eta_1^{(0)}, \eta_2^{(0)})$ of $\Gamma_{w^{(0)}}$ at $(w^{(0)}, 1)$ should satisfy

$$
(M_1, M_2)\begin{pmatrix}\eta_1^{(0)}\\ \eta_2^{(0)}\end{pmatrix} = 0,
$$

where $\eta_1^{(0)} \in \Re^{n+m+1+l}$ and $\eta_2^{(0)} \in \Re$. By direct computation, we have $\eta_1^{(0)} = -M_1^{-1}M_2\eta_2^{(0)}$. The determinant in (3.19) is of the form

$$
\begin{aligned}
\left|\begin{matrix}\nabla H_{w^{(0)}}(w^{(0)}, 1)\\ \eta^{(0)\mathrm{T}}\end{matrix}\right| &= \begin{vmatrix} M_1 & M_2\\ -M_2^{\mathrm{T}}M_1^{-\mathrm{T}} & 1\end{vmatrix}\eta_2^{(0)}\\
&= \begin{vmatrix} M_1 & M_2\\ 0 & 1 + M_2^{\mathrm{T}}M_1^{-\mathrm{T}}M_1^{-1}M_2\end{vmatrix}\eta_2^{(0)}\\
&= |M_1|\left(1 + M_2^{\mathrm{T}}M_1^{-\mathrm{T}}M_1^{-1}M_2\right)\eta_2^{(0)}. \qquad (3.20)
\end{aligned}
$$

By the definition of M_1, we have

$$
\begin{aligned}
|M_1| &= \begin{vmatrix}
I & 0 & 0 & 0\\
0 & I & 0 & 0\\
u^{(0)}\nabla_x g(x^{(0)}, y^{(0)})^{\mathrm{T}} & u^{(0)}\nabla_y g(x^{(0)}, y^{(0)})^{\mathrm{T}} & g(x^{(0)}, y^{(0)}) & 0\\
0 & V^{(0)}\nabla h(y)^{(0)})^{\mathrm{T}} & 0 & h(y^{(0)})
\end{vmatrix}\\
&= \left|g(x^{(0)}, y^{(0)})\right|\begin{vmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & V^{(0)}\nabla h(y^{(0)})^{\mathrm{T}} & h(y^{(0)})\end{vmatrix}\\
&= \left|g(x^{(0)}, y^{(0)})\right|\begin{vmatrix} I & 0\\ V^{(0)}\nabla h(y^{(0)})^{\mathrm{T}} & h(y^{(0)})\end{vmatrix}\\
&= \left|g(x^{(0)}, y^{(0)})\right|\left|h(y^{(0)})\right|. \qquad (3.21)
\end{aligned}
$$

By (3.21), (3.20) can be rewritten as

$$\begin{aligned}\left|\begin{array}{c}\nabla H_{x^{(0)}}(x^{(0)},1)\\ \eta^{(0)\mathrm{T}}\end{array}\right| &= |M_1|\left(1+M_2^{\mathrm{T}}M_1^{-\mathrm{T}}M_1^{-1}M_2\right)\eta_2^{(0)}\\ &= \left|g(x^{(0)},y^{(0)})\right|\left|h(y^{(0)})\right|\\ &\quad\cdot\left(1+\left(M_1^{-1}M_2\right)^{\mathrm{T}}M_1^{-1}M_2\right)\eta_2^{(0)}.\end{aligned}$$

Note that $g(x^{(0)},y^{(0)})<0$, $h(x^{(0)})<0$, $1+\left(M_1^{-1}M_2\right)^{\mathrm{T}}M_1^{-1}M_2>0$ and $\eta_2^{(0)}$ should be negative since initially we plan to move along the path $\Gamma_{x^{(0)}}$ by decreasing μ, and hence the sign of

$$\left|\begin{array}{c}\nabla H_{w^{(0)}}(w^{(0)},1)\\ \eta^{(0)\mathrm{T}}\end{array}\right|$$

is $(-1)^{l+2}$. This completes the proof of lemma. □

Associating with lemma 3.3, a new predictor point $\hat{z}^{(k+1)}$ can be generated at point $z^{(k)}$ along the direction $\eta^{(k)}$, i.e. $\hat{z}^{(k+1)}=z^{(k)}+h\eta^{(k)}$ for some small steplength $h>0$. Next, we may make a corrector step. Setting $M_k=\nabla H_{w^{(0)}}(w^{(k)},\mu_k)$, the matrix

$$M_k^{+}=M_k^{\mathrm{T}}(M_kM_k^{\mathrm{T}})^{-1}\in\Re^{(n+m+1+l+1)\times(n+m+1+l)}$$

is the Moore-Penrose inverse of M_k. The corrector phase then tries to identify a point $(w(s),\mu(s))$ on the path near to $\hat{z}^{(k+1)}$; that is used to define the next iterate $z^{(k+1)}=(w(s_{k+1}),\mu(s_{k+1}))$. The corrector step is usually carried out by a version of Newton's method that uses the Moore-Penrose inverse of M_k, starting with $\hat{z}^{(k+1)}$ and proceeding until $||H_{w^{(0)}}(w,\mu)||$ is approximately zero. If a predictor step produces a point that cannot be sufficiently corrected or fewer iterations of the corrector step, a revised, more conservative steplength adaptation is made by some steplenth strategies, and then the corrector step is attempted again. The following pseudocode shows the basic steps of a generic predictor–corrector method.

Algorithm 3.1 (*Generic PC method for approximate solution of* (3.3))
Step 0 *Give* $\mu_0=1$, *an initial point* $y^{(0)}=(w^{(0)},\mu_0)$, *step length* $h_0>0$ *and two small positive numbers* ε_1, μ_ε, $k:=0$.
Step 1 *Compute a predictor point* $(w^{(k+1,0)},\mu_{k+1,0})$:

(a) *Compute a unit tangent vector* $\xi^{(k)}\in\Re^{n+m+1+l+1}$;
(b) *Determine the direction* $\eta^{(k)}$ *of predictor step*:

If the sign of the determinant

$$\left|\begin{array}{c}\nabla H_{w^{(0)}}(w^{(k)},\mu_k)\\ \xi^{(k)\mathrm{T}}\end{array}\right|$$

is $(-1)^{l+2}$, *then* $\eta^{(k)}=\xi^{(k)}$, *else* $\eta^{(k)}=-\xi^{(k)}$.
(c) *Set* $(w^{(k,0)},\mu_{k,0})=(w^{(k)},\mu_k)+h_k\eta^{(k)}$.

Step 2 *Compute a corrector point* $(w^{(k+1)},\mu_{k+1})$:

$$(w^{(k+1,j)},\mu_{k+1,j})=(w^{(k,j-1)},\mu_{k,j-1})-M_{k,j-1}^{+}H_{w^{(0)}}(w^{(k,j-1)},\mu_{k,j-1}),\ j=1,2,\ldots$$

until $||H_{w^{(0)}}(w^{(k+1,j)}, \mu_{k+1,j})|| \leq \varepsilon_1$, *and set*

$$(w^{(k+1)}, \mu_{k+1}) = (w^{(k+1,j)}, \mu_{k+1,j}).$$

Step 3 *If* $\mu_{k+1} \leq \mu_\varepsilon$, *then stop, else choose a new steplength* $h_{k+1} > 0$. $k := k + 1$, *and go to Step 1.*

4 Preliminary numerical results

To illustrate the computational implementation of the proposed algorithm, several test problems in this section are listed in the following and solved by previous homotopy algorithm by using the software of MATLAB6.5 on a PC with Intel4 CPU PI 2.0 MHZ and DOS6.22. The numerical results given here are obtained by using the above Euler–Newton procedure (3.1). In all computation, the termination criterion in corrector step is $\varepsilon_1 = 1.0e-010$ and the setting for the parameter is $h_k \equiv h_0 = 0.01$. For convenience, some multipliers appearing in each example are the same as

$$u^{(0)} = 1.0, \quad v^{(0)} = (1.0, 1.0)^{\mathrm{T}}.$$

Moreover, numerical results are computed by short-type precision of matlab 6.5' language, and are reported in the tables. The Table 1 includes the initial point, approximate solution and value-evaluation of the objective function.

Example 4.1 [15] $f(x) = 2.25\exp(x_1) + \exp(x_2), g(x, y) = y - \exp(x_1 + x_2), \mathcal{Y} = [0, 1]$. $x^* = (-4.05e-01, 4.05e-01)$. $y^* = 1.0000$.

Example 4.2 [10] $f(x) = 1.21\exp(x_1) + \exp(x_2), g(x, y) = y - \exp(x_1 + x_2), \mathcal{Y} = [0, 1]$. $x^* = (-9.53e-02, 9.53e-02)$. $y^* = 1.0000$.

Example 4.3 [9] $f(x) = 2x_1 + x_2$, $g(x, y) = -y^2 + y - yx_1 + (y - 1)x_2$, $\mathcal{Y} = [0, 1]$. $x^* = (1/9, 4/9)$. $y^* = 0.6666$.

Table 1 Results for PC algorithm

Example	l	$x^{(0)}$	$y^{(0)}$	$\bar{x}$	$\bar{y}$	$f(\bar{x})$	μ_ϵ
4.1	2	(1.0, 1.0)	0.50	(−0.4057, 0.4053)	1.0000	8.1525e − 011	1.0e − 03
		(1.0, 1.0)	0.90	(−0.4057, 0.4053)	1.0000	9.6164e − 011	
		(2.0, 3.0)	0.50	(−0.4036, 0.4074)	1.0000	9.9261e − 011	
		(2.0, 3.0)	0.70	(−0.4034, 0.4075)	1.0000	4.6621e − 011	
4.2	2	(1.0, 1.0)	0.50	(−0.0941, 0.0965)	0.9998	8.9495e − 011	1.0e − 03
		(1.0, 1.0)	0.60	(−0.0966, 0.0941)	1.0002	5.9523e − 011	
		(2.0, 1.0)	0.50	(−0.0964, 0.0944)	1.0001	9.8645e − 011	
		(3.0, 1.0)	0.50	(−0.0974, 0.0934)	1.0000	9.9828e − 011	
4.3	2	(1.0, 0.0)	0.50	(0.1111, 0.4443)	0.6667	2.7137e − 011	1.0e − 03
		(1.0, 0.0)	0.75	(0.1107, 0.4448)	0.6668	6.2663e − 011	
		(0.0, 1.0)	0.75	(0.1110, 0.4446)	0.6664	2.1618e − 011	
		(2.0, 1.0)	0.75	(0.1113, 0.4445)	0.6666	4.0670e − 011	
		(4.0, 9.0)	0.50	(0.1110, 0.4439)	0.6665	3.3400e − 011	

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